Linear-Angle Solutions to the Optimal Rocket Steering Problem

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Introduction

THE optimal guidance of a launch vehicle from a planetary surface into planetary orbit stands out as one of the most successful applications of optimal control theory/calculus of variations. The theoretical foundations of this problem were well developed during the 1950's by a number of investigators, although some isolated and partial work has been done earlier, for example by R. M. Goddard and K.E. Tsiolkovsky in 1903.

Lawden's[1] statement of the solution remains as one of the most elegant:

$$\ddot{p}=(p\cdot\nabla)g \tag{1}$$

where p is Lawden's primer vector and is in the direction in which the thrust should be applied, and g is the local gravity field. Equation (1) is one of those deceptively elegant equations which, behind an innocent exterior, conceal a maze of intricacies (to which measure they no doubt owe much of their elegance). Since g is a function of position x, the terms involving the derivatives of $g(x)$ are generally not known until the trajectory has been established; in other words, until the problem has been solved.

Attempts to separate out the dependence upon x have generally been minimally successful, except for one special important case in which g is independent of x, or more generally, no more than a linear function of x. In the case of constant g, the solution is well known to be given by the linear tangent law

$$\tan\chi=b+ct \tag{2}$$

where χ is the space-fixed pitch angle of the thrust vector in a suitable coordinate system, t is time, and b and c are constants.

Attempts to generalize Eq. (2) to a Newtonian (inverse-square) gravitational field have, as mentioned, generally met with minimal success.

One result of this has been the development of a field known as "parameterized guidance." In parameterized guidance, the objective is to find approximate solutions to Eq. (1) containing sufficient arbitrary parameters that, by assigning the proper values to these parameters, a steering history can be formulated which results in the acquisition of the desired end conditions.

Different guidance modes have arisen depending upon which approximation is selected. One natural candidate is Eq. (2), and this has been incorporated by Perkins[2] into what is known as the "linear tangent steering mode." Other investigators have noted that many numerically integrated examples seem to exhibit a pitch profile much more linear than would be expected based on the linear tangent law. They have suggested that perhaps a linear-angle (as opposed to linear-tangent) law might be more appropriate for trajectories of actual interest. One such implementation by Smith[3] is known as the "iterative guidance mode" and has been used very successfully on the Saturn launch vehicle program. A study performed by TRW[4] suggests that for large thrust arcs, linear-angle steering seems to be superior to linear-tangent steering. A summary and evaluation of other guidance modes, in addition to the above two, is contained in Ref. 5.

The very nearly linear nature of many solutions and the great success of the "iterative guidance mode" suggest that linear-angle solutions to Eq. (1) should be investigated, which is discussed in the next section.

Exact Linear Solutions

In the previous section it was pointed out that ascent-to-orbit pitch profiles are often very nearly linear. This suggests that with a small change in terminal conditions and, possibly, acceleration profile, exact linear solutions to the optimal steering problem may be found. However, this turns out not to be the case. Linear solutions do exist, but the nature of the trajectories is quite different from the ascent-to-orbit type. We now investigate this and begin with Lawden's equation (Eq. 1) transformed to a uniformly rotating coordinate system:

$$\ddot{p}=\frac{\partial^2 p}{\partial t^2}+2\eta\times\frac{\partial p}{\partial t}-\eta^2 p=(p\cdot\nabla)g \tag{3}$$

where η is the angular velocity vector and is normal to p, and $\eta^2=|\eta|^2=$constant.

The explicit form of Eq. (3) is:

$$\begin{bmatrix}\ddot{p}_1-2\eta\dot{p}_2-\eta^2p_1\\ \ddot{p}_2+2\eta\dot{p}_1-\eta^2p_2\end{bmatrix}=(p\cdot\nabla)g=\begin{bmatrix}p_1g_{11}+p_2g_{12}\\ p_1g_{21}+p_2g_{22}\end{bmatrix} \tag{4}$$

where $g_{11}=\partial g_1/\partial x_1$, $g_{12}=g_{21}=\partial g_1/\partial x_2$, etc.

If p is in uniform rotation, then by choosing η to be this rotation rate, we may choose the X_1 axis to be parallel to p. This means that $p_2\equiv 0$. Equation (4) therefore reduces to

$$\ddot{p}_1-\eta^2p_1=p_1g_{11} \qquad 2\eta\dot{p}_1=p_1g_{21} \tag{5a,b}$$

Differentiate Eq. (5b) and use the result to obtain

$$2\eta\frac{\mathrm{d}}{\mathrm{d}t}g_{12}+(g_{12})^2-4\eta^4-4\eta^2g_{11}=0 \tag{6}$$

Now, let ψ be the angle between the local horizontal and the primer vector as shown in Fig. 1. ($\psi=-\pi/2$ is straight down along the gravity vector.) Then,

$$g_{11}=\omega^2\left(\frac{1}{2}-\frac{3}{2}\cos 2\psi\right) \qquad g_{12}=\omega^2\frac{3}{2}\sin 2\psi \qquad \omega^2=\frac{\mu}{R^3} \tag{7}$$

where ω is the local Schuler angular velocity, i.e., the angular velocity of a circular orbit. Differentiating g_{12} and substitution into Eq. (6) yields:

$$2\eta\omega^2[3\cos(2\psi)\dot{\psi}-(9/2)\sin(2\psi)\dot{R}/R]$$

$$+(9/4)\omega^4\sin^2(2\psi)-4\eta^4-2\eta^2\omega^2+6\eta^2\omega^2\cos 2\psi=0 \tag{8}$$

Suppose now that the trajectory passes through circular orbit conditions. At that instant we must have $\dot{R}=0$ and $\dot{\theta}=-\omega$, where θ is the polar angle, as in Fig. 1. (The minus sign is for consistency with Fig. 1 which shows the trajectory moving from left to right; hence $\dot{\theta}$ must be negative.) Also since, in

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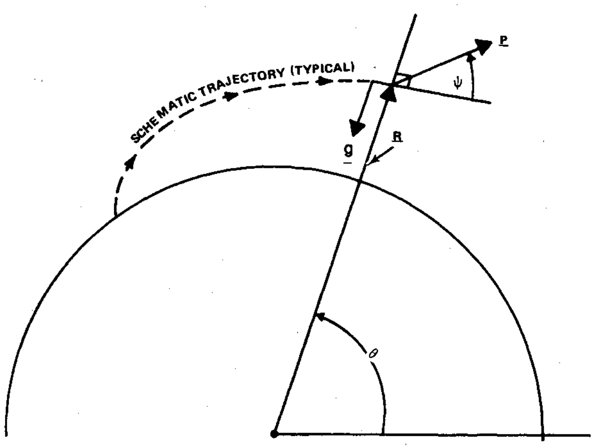

Fig. 1 Trajectory and guidance geometry.

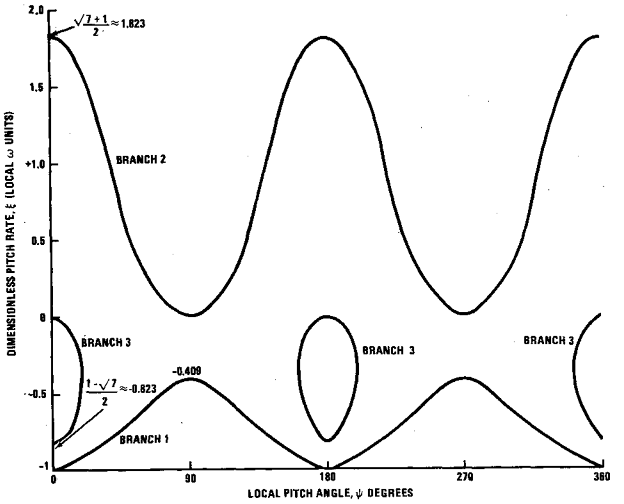

Fig. 2 Allowable (constant) pitch rates as a function of local pitch angle.

general, $\dot{\psi}=\eta-\dot{\theta}$, then at the instant of circular conditions, we must have

$$\dot{\psi}=\eta+\omega$$

Substituting into Eq. (8) and defining $\xi=\eta/\omega$, we obtain

$$2\xi^4+(1-6\cos2\psi)\xi^2-3\cos(2\psi)\xi-(9/8)\sin^2 2\psi=0 \tag{9}$$

as an equation which relates the pitch rate $(\eta=\xi\omega)$ and the pitch angle in a local reference frame (ψ) at the instant of passing through circular orbit conditions. This equation is numerically investigated in Fig. 2, where ξ is shown as a function of ψ. The range of ξ is $[-1, 1.823]$.

Now it is well known that for Earth surface-to-orbit trajectories, we obtain ξ's of about -2 (assuming the convention that the motion is in the $+x$ direction, as we have done here). It follows that (exactly) uniformly rotating primer vectors do not belong to the family of surface-to-orbit trajectories.

Further Necessary Conditions

Figure 2 shows that for every ψ there exist at least two values of η for which χ is at an inflection point $(\ddot{\chi}=\dot{\eta}=0)$. However, while Eq. (9), on which Fig. 2 is based, provides a necessary condition for $\dot{\eta}=0$, we have no guarantee that a short time later Eq. (9) will still be satisfied by the same η. This, of course, is required if η is to be truly constant. Thus, we must determine whether there exists any acceleration level

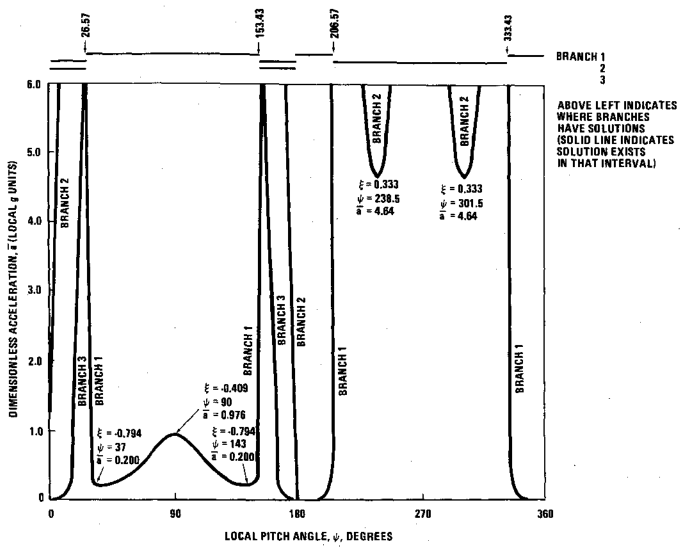

Fig. 3 Required acceleration level for constant pitch rate.

which when applied along p will keep the trajectory on the curve of Eq. (9). Anticipating ahead a little, we shall find there are four possibilities:

1) The angle $\psi+\pi$, rather than ψ, is the angle at which thrust should be applied.

2) The primer vector corresponds to a natural (i.e., force-free/conic-section) motion. In this case no extragravitational acceleration is required, nor can any, save infinitesimal, be tolerated. These are essentially "low-thrust" solutions.

3) The required acceleration is infinite. These points are therefore merely inflection points. All thrusting maneuvers are impulsive.

4) There exists a unique acceleration level at which the relationship $\ddot{\chi}=0$ is preserved. This is of course the most interesting case.

We now proceed to derive the equation for the acceleration level discussed in item 4 above. The most efficient method is to differentiate Eq. (8) and substitute for $\ddot{R}$ and $\ddot{\psi}$ by means of the equations of motion:

$$\ddot{R}-\dot{\theta}^2R=g+a\sin\psi \qquad R\ddot{\theta}+2\dot{R}\dot{\theta}=-a\cos\psi \tag{10}$$

As previously, we shall restrict ourselves to the case in which the trajectory is at the instant of consideration passing through circular orbit conditions. In this case, by virtue of $\dot{R}=0$ and $\dot{\theta}=-\omega$, Eqs. (10) simplify to

$$\ddot{R}=a\sin\psi \qquad \ddot{\theta}=-a\cos\psi/R$$

Also, in differentiating Eq. (8), we use the following relations:

$$\dot{\eta}=\dot{R}=\dot{\omega}=0 \qquad (\dot{R}=0\rightarrow\dot{\omega}=0)$$

$$\dot{\psi}=\eta+\omega \qquad \ddot{\psi}=-\ddot{\theta}=a\cos\psi/R$$

The equation which results is

$$\bar{a}=\frac{a}{g}=\frac{2\sin\psi(1+1/\xi)(8\xi^2+4\xi-3\cos2\psi)}{(5\cos2\psi-3)} \tag{11}$$

Figure 3 contains a plot of $\bar{a}$. Several function branches appear corresponding to the branches of Fig. 2. There are several interesting features. First of all is the importance of the angle ψ^* given by the vanishing of the denominator in Eq. (11): $\psi^*\approx26.57$ deg, 153.43 deg, etc. At these angles there exist only impulsive solutions $(a=\infty)$, and this is due to the fact that these are inflection points, as discussed in item 3 above. Secondly, note that $\psi=0$ and $\psi=180$ yield only low-thrust solutions. Finally, we note that except for the above six points (two low-thrust, four impulsive) and one additional singularity at 270 deg there exists a finite solution for $\bar{a}$ at every ψ. However, on some intervals such as [206 deg, 227 deg] the acceleration level is above 6.0 g's, and for all practical purposes these must be considered impulsive solutions.

Another interesting feature is the existence of four minima (discounting the zero minima) and one maximum (the one 270 deg is infinite). What is particularly interesting is that the maximum at 90 deg has a value very nearly equal to, but not exactly equal to, unity. (The value is about 0.976.) The two smaller minima have values around 0.2, while the larger minima have values of about 4.64. Again, this last value is sufficiently large to be considered an impulsive solution. Thus the two minima at $\psi=37$ and $\psi=143$ and the single maximum at 90 deg are the three most interesting points. Detailed exploration of the trajectories corresponding to these is left to another paper, but it appears that $\psi=90$ deg is associated with the circularization of a straight-in parabolic trajectory ($R_p=0$, $R_a=\infty$), while the two minima are associated with Hohmann-type maneuvers.

References

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Optimal Low-Thrust Maneuvers Near the Libration Points

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Introduction

As has been previously reported[1] an investigation was made of the minimum propellant optimal maneuvers near the libration points in the Earth-moon system of a space vehicle equipped with a low-thrust propulsion installation. The study was made only for collinear libration points on the basis of some approximate solutions of the differential equations of the extremals ($\omega=0$). The present paper carries the work further and gives this study a complete form on the basis of some rigorous solutions both for collinear and equidistant libration points ($\omega\neq0$).

The system of units and the notations in this Note are those used in Ref. 1.

Variational Problem

In Ref. 1 it was shown that generally the variational problem of the minimum propellant optimal maneuvers near the libration points is one of extremum with constraints. On introducing Lagrange's multipliers λ_1, λ_2, λ_3, λ_4, and reducing the problem of extremum with constraints to one of extremum without constraints, after performing the calculations we get the following system of differential equations of the extremals:

$$\frac{d\xi}{dt}=V_\xi \qquad \frac{d\eta}{dt}=V_\eta \tag{1a}$$

$$\frac{dV_\xi}{dt}=2\omega V_\eta+K_1\xi+K_3\eta+a_\xi \tag{1b}$$

$$\frac{dV_\eta}{dt}=-2\omega V_\xi+K_2\eta+K_3\xi+a_\eta \tag{1c}$$

$$\frac{d\lambda_1}{dt}=-K_1\lambda_2-K_3\lambda_4 \qquad \frac{d\lambda_2}{dt}=-\lambda_1+2\omega\lambda_4 \tag{1d}$$

$$\frac{d\lambda_3}{dt}=-K_3\lambda_2-K_2\lambda_4 \qquad \frac{d\lambda_4}{dt}=-2\omega\lambda_2-\lambda_3 \tag{1e}$$

where the quantities K_1, K_2, K_3 are given in Ref. 1; and the algebraic equations:

$$2a_\xi-\lambda_2=0 \qquad 2a_\eta-\lambda_4=0 \tag{2}$$

Exact Solutions for the Collinear and Equidistant Libration Points

In order to integrate Eqs. (1a-1c), it is necessary first to know the functions a_ξ and a_η which, taking account of Eq. (2), can be obtained by integrating Eqs. (1d) and (1e).

The differential equation system [Eqs. (1d) and (1e)] has the characteristic equation

$$\Lambda^4+b\Lambda^2+c=0 \tag{3}$$

where

$$b=4\omega^2-K_1-K_2 \qquad c=K_1K_2-K_3^2$$

which can have either two real and two imaginary roots or all four roots imaginary.

We consider first the case where two roots are real and two are imaginary, the case of all roots imaginary being obtained from the first by particularization.

Let

$$\Lambda_{1,2}=\pm p \qquad \Lambda_{3,4}=\pm iq \tag{4}$$

where

$$p^2,q^2=\tfrac{1}{2}(\mp b+\sqrt{b^2-4c})$$

Using the current method of integration, the solution of Eqs. (1d) and (1e) can be set after some calculation under the form

$$\lambda_1=D_1\cosh pt+D_2\sinh pt+D_3\cos qt+D_4\sin qt \tag{5a}$$

$$\lambda_2=(A_1D_1+A_2D_2)\cosh pt+(A_2D_1+A_1D_2)\sinh pt + (A_3D_3-A_4D_4)\cos qt+(A_4D_3+A_3D_4)\sin qt \tag{5b}$$

with λ_3 and λ_4 obtained by replacing A_k with B_k and A_k with C_k, respectively ($k=1,2,3,4$). The quantities A_k, B_k, and C_k are given in Appendix A.

From the above expressions we obtain

$$a_\xi=\tfrac{1}{2}\lambda_2 \qquad a_\eta=\tfrac{1}{2}\lambda_4 \tag{6}$$

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